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Constrained Relative Attitude Determination for Two-Vehicle Formations

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This paper studies constrained relative attitude determination of a formation of two vehicles. A deterministic solution for the relative attitude between the two vehicles with line-of-sight observations between them and a common object observed by both vehicles is presented. The constraint may either be a planar constraint or triangle constraint of the observed vectors. The differences between both are shown. Either constraint allows for a solution without having to know the location of the common object. The presented solution with the constraint represents the minimum number of measurements required to determine the relative attitude and no ambiguities are present. To quantify the performance of the algorithm the covariance of the attitude error is derived using a linearized error model from a least-squares point of view. A sensitivity analysis is also performed in order to assess how out-of-plane observations affect the overall solution. Simulation results are provided to assess the performance of the proposed new approach.

I. Introduction

FORMATION flying employs multiple vehicles to maintain a specific relative attitude/position, either in a statically or a dynamically closed trajectory. Here relative is defined as being between two vehicles. Relative information is generally needed to maintain formation attitude through control. Applications are numerous involving all types of vehicles, including land (robotics [1]), sea (autonomous underwater vehicles [2]), space (spacecraft formations [3]) and air [uninhabited air vehicles (UAVs) [4]] systems. As a specific example, UAV technology has gained widespread use in recent years for military and civilian applications. More than 30% of all the Air Force's reconnaissance aircraft are now pilotless [5]. Relative vehicle navigation will be required to maintain formation topologies for a variety of reasons. For example flying UAVs in close formation can simulate an aircraft with a large aspect ratio, reducing the induced drag of each vehicle in the formation and providing an improvement in overall efficiency [6]. Equally promising is the improvements that UAV formation flight offers over current distributed sensing technology. Applications include surface-to-air missile jamming [7], radar deception [8], synthetic aperture radar interferometry [9], and surveillance and reconnaissance. To achieve the desired level of accuracy for this sensing application not only must the UAVs avoid collision with each other and other obstacles, but they must maintain the desired configuration. Cooperative UAV formation flight requires precise relative position and attitude for formation control and coordination.

Most inertial navigation systems used for UAVs incorporate the Global Positioning System (GPS) along with inertial measurement units providing both inertial position and attitude. If relative information is required then these measurements must be converted to relative coordinates. Although GPS can be used to provide relative information using pseudolites, GPS and GPS-like signals are susceptible to interference and jamming, among other issues. Therefore developing GPS-less navigation systems is currently an active area of research [10]. Recent research concerning vision-based navigation (VISNAV) for UAVs indicates that relative navigation can be achieved using camera-based images. Line-of-sight (LOS) vectors between vehicles in formation can be used for relative navigation and in particular relative attitude determination. Reference [11] implements an extended Kalman filter to estimate the relative position and attitude of two air vehicles using multiple LOS measurements between them along with other onboard measurements from gyros and accelerometers. This approach has the advantage of not relying on external sensors but may require considerable onboard computations. Computing the relative attitude directly without filtering for the two-vehicle formation using LOS information between them can offer computational efficiency without reliance on filter convergence issues because point-by-point solutions are possible with deterministic methods.

Many algorithms have been published to determine the attitude from two or multiple unit vectors, the most widely used of which are the TRIAD [12] and QUEST [13] algorithms. When more than the minimal set of vector observations is used to determine the attitude an optimal solution is obtained by minimizing an appropriate cost function, which was first introduced as the well-known Wahba problem [14]. A purely deterministic solution for the attitude involves one direction and one angle or three angles but this case is shown to have a discrete ambiguity [15], which needs further information to resolve. The advantages of a deterministic solution are:

1) Since the minimal scalar measurements are used there is no need to minimize a cost function.

2) Any deterministic algorithm will provide an optimal solution.

3) A deterministic solution is very attractive for hardware implementation in real time.

Using a set of LOS observations between vehicles in a three-vehicle formation has been shown to offer a deterministic solution [16], which is not possible if each vehicle is considered separately. The observability of this relative attitude solution depends on both vehicle geometry and sensor location. It is well known that the rotation around a unit vector is unobservable when that unit vector is the only observation used for attitude determination. Reference [16] shows that having only one LOS set between each of the individual

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vehicles provides sufficient information to determine all relative attitudes in a three-vehicle system. An unobservable case arises when all vectors are in the same plane, e.g., they form a triangle. To overcome this problem the sensor/emitter location of one vehicle must not be in the same plane as that formed by the sensors of the other two vehicles. This work extends the previous result to a two-vehicle formation with a common observed object, which can be another vehicle or a landmark, by applying a parametric constraint to the attitude solution. This constraint is based on assuming that a triangle set of observations is given. In the work of [16] this issue causes problems in the solution, while in the present work this constraint is forced to be true and hence relieves the arising difficulties. This results in a deterministic solution for the relative attitude with no ambiguity and no observability issues.

The triangle scenario does reflect a realistic physical situation. For example, this occurs naturally when two UAVs have a common LOS between them and measure some common object other than each other, which forms a triangle of LOS observations. It is important to note that no information on the location of the object is required in the presented solution, only the fact that both vehicles observe the common object. This constitutes a significant departure from standard navigation or attitude approaches that use known objects or landmarks. The triangle constraint is used to determine a solution. However, due to sensor misalignments and/or noise in the measurements the actual LOS observations will not form a perfect triangle. This error will be studied by deriving an analytical expression of the error sensitivity for out-of-plane vectors.

The triangle/planar constraint has been used in stereo-vision (binocular stereopsis) and is known as an "epipolar" constraint [17]. The epipole is the point of intersection of the line joining the optical centers, i.e., the baseline, with the image plane. The main purpose of binocular stereopsis is to triangulate the location of some object using two cameras. However, before this is accomplished, determination of a set of intercamera parameters must first be done. Epipolar geometry is used to determine the "essential matrix" for the calibrated problem or the "fundamental matrix" for the uncalibrated problem. The essential matrix contains both the relative position and orientation (extrinsic parameters) between the two cameras, while the fundamental matrix incorporates other calibration terms (intrinsic parameters) in addition to the extrinsic parameters. To determine either the essential or fundamental matrix multiple feature points must be obtained. The work presented here is different than binocular stereopsis in that only relative attitude is considered, which leads to an approach that requires only one feature point, i.e., the common object.

The organization of this paper is as follows. First, a discussion of the nature of the problem is given and two approaches for applying constraints are discussed. Then, the sensor model for the LOS measurements is reviewed. Next, the two-vehicle formation relative attitude determination solution is shown using two constraint approaches. A relative attitude error covariance is derived using a linearized error model. Then, a sensitivity expression to out-of-plane deflections is derived. Finally, simulation results are shown for both a static and dynamic formation.

II. Problem Definition

Noting Fig. 1, the case of two vehicles in relative formation flight is considered. Each vehicle has a local separate body frame denoted by $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. The inertial attitude of each vehicle is given by $A^{\mathcal{I}}_{\mathcal{B}_1}$ and $A^{\mathcal{I}}_{\mathcal{B}_2}$, respectively, where $\mathcal{I}$ denotes inertial frame. The relative attitude describing the mapping from $\mathcal{B}_1$ to $\mathcal{B}_2$ can be written as $A^{\mathcal{B}_2}_{\mathcal{B}_1} = A^{\mathcal{I}^T}_{\mathcal{B}_2} A^{\mathcal{I}}_{\mathcal{B}_1}$.

Each vehicle observes a LOS from itself to the other vehicle in the formation as well as a common object such as a landmark. If any of the two-vector algorithms described in Sec. I are to be used, then it is necessary to know the components of the two-vector observations in both frames. The classical star camera problem can be solved using these algorithms assuming that there is no parallax between observations made in each frame. This assumption is highly accurate because the distance to the reference stars is large in comparison to the baseline distance between each frame. In the case of the UAV example this assumption is not always valid because the distance between the reference object and the vehicle may be comparable to the distance between the two frames. Therefore the parallax issue needs to be resolved by an origin transformation. This typically requires associated range information which introduces more error into the algorithm [16]. Here it is assumed that parallel beams between the vehicles exists, so that range information is not required. This can be accomplished by employing a feedback device into the overall vehicle-to-vehicle sensor system. The LOS observation vectors used in this paper are denoted by the following, as shown by Fig. 2:

1) The vector $\mathbf{w}_1$ is the vector from the $\mathcal{B}_2$ vehicle frame to the $\mathcal{B}_1$ vehicle frame expressed in $\mathcal{B}_2$ coordinates.

2) The vector $\mathbf{v}_1$ is the vector from the $\mathcal{B}_2$ vehicle frame to the $\mathcal{B}_1$ vehicle frame expressed in $\mathcal{B}_1$ coordinates. Note that in actual practice the negative of the vector is measured by the sensor, as shown by Fig. 1.

3) The vector $\mathbf{w}_2$ is the vector from the $\mathcal{B}_2$ vehicle frame to the common object expressed in $\mathcal{B}_2$ coordinates.

4) The vector $\mathbf{v}_2$ is the vector from the $\mathcal{B}_1$ vehicle frame to the common object expressed in $\mathcal{B}_1$ coordinates.

The observations $\mathbf{w}_1$ and $\mathbf{v}_1$ can be related to each other through the attitude matrix mapping:

$$\mathbf{w}_1 = A^{\mathcal{B}_2}_{\mathcal{B}_1} \mathbf{v}_1 \tag{1}$$

Note that all vectors, $\mathbf{w}_1$, $\mathbf{v}_1$, $\mathbf{w}_2$ and $\mathbf{v}_2$, are required to be unit vectors.

It is well known that using a single pair of LOS vectors between the two vehicles does not provide enough information for a complete three-axis relative attitude solution. In particular, to determine the full attitude the rotation angle about the LOS direction must also be determined. The following property of the attitude matrix is now considered: $((A^{\mathcal{B}_2}_{\mathcal{I}})^T \ \mathbf{w}_2)^T (A^{\mathcal{B}_1}_{\mathcal{I}})^T \mathbf{v}_2 = \mathbf{w}_2^T A^{\mathcal{B}_2}_{\mathcal{I}} (A^{\mathcal{B}_1}_{\mathcal{I}})^T \ \mathbf{v}_2 = \mathbf{w}_2^T A^{\mathcal{B}_2}_{\mathcal{B}_1} \mathbf{v}_2$, which means that the attitude matrix preserves the angle between vectors. This yields the following equation:

$$d = \mathbf{w}_2^T A^{\mathcal{B}_2}_{\mathcal{B}_1} \mathbf{v}_2 \tag{2}$$

where d is the cosine of the angle between the two LOS vectors to the common object and $A^{\mathcal{B}_2}_{\mathcal{B}_1}$ is denoted as A from this point forward. In [16] this angle is determined from two LOS vectors observed to and from a third vehicle in a three-vehicle formation. This requires an extra LOS vector between the two vehicles and the third vehicle, which is not the case here, however, as will be seen.

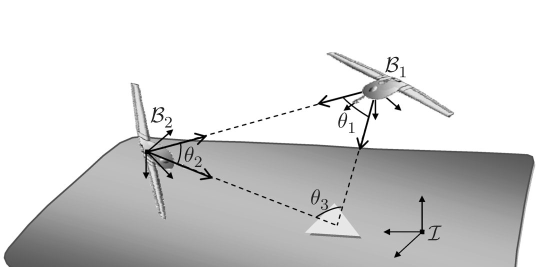

Fig. 1 Vehicle formation.

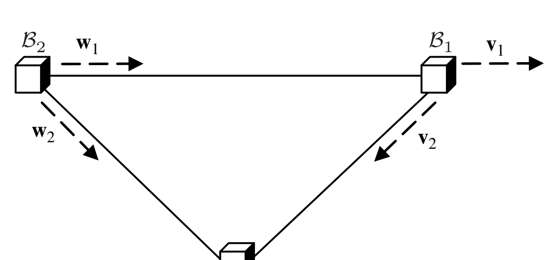

Fig. 2 Observation geometry.

Solving the preceding equations yields a deterministic solution for the relative attitude between the two frames. But if these equations are to be used directly, then the angle between $\mathbf{w}_2$ and $\mathbf{v}_2$ must be observed by the third object in the formation. Since the observations constitute the legs of a triangle (see Fig. 1) and the angles in a triangle must add up to π, then the angles are not independent of each other. If two of the angles are known, then the third angle is automatically known. This third angle can be angle between the LOS to the common object observed from both frames, and therefore a solution to the full attitude can be determined for this observation geometry by constraining the observations to form a triangle. Since for the observation geometry considered here all observation vectors lie on a common plane, then a plane constraint can also be applied to solve for this rotation angle. First the LOS vector between the two frames can be aligned through an initial rotation, then a rotation angle about this direction can be found such that when this rotation is applied the angle between the observations add up to π or the vectors lie on a common plane. The third reference object in the formation does not need to communicate its LOS observations to the other two vehicles for the solution of their relative attitude. Therefore a very powerful conclusion can be made from this observation: choice of the common third object in the formation is arbitrary and can be any common reference point, with unknown position, when the geometrical condition is applied.

Now a condition is applied that is present in the observations by means of a constraint, i.e., the form of the geometry considered is known: the observation vectors constitute the legs of a triangle or they lie on a common plane. Two constraints will be considered, one where the observation vectors are constrained to lie on the same plane and another where the angles between the LOS vector are constrained to add up to π. These two constraints are referred to as the planar and triangle constraints, respectively.

The planar constraint can be simply written as

$$0 = \mathbf{w}_2^T[\mathbf{w}_1\times]A\mathbf{v}_2 \tag{3}$$

where the matrix $[\mathbf{w}_1\times]$ is the cross product matrix. The definition of this matrix for a general 3×1 vector $\boldsymbol{\alpha}$ is given by

$$[\boldsymbol{\alpha}\times] \equiv \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix} \tag{4}$$

Note that this planar constraint is a less rigorous constraint than the triangle constraint because if the first observation equation in Eq. (1) is satisfied, then there are two possible configurations that satisfy this constraint: one being the actual observation geometry and the other where the $A\mathbf{v}_2$ vector is rotated by 180 deg from the true configuration.

A more rigorous constraint is where the angles between the vectors are constrained. To determine the constraint function for the triangle condition the following is used:

$$\theta_3 = \pi - \theta_1 - \theta_2 \tag{5}$$

where the angles θ_1, θ_2 and θ_3 are defined in Fig. 1. Taking the cosine of each side leads to

$$\cos(\theta_3) = \cos(\pi - \theta_1 - \theta_2) \tag{6}$$

and

$$\cos(\theta_3) = \cos(\theta_2)\cos(\pi - \theta_1) + \sin(\theta_2)\sin(\pi - \theta_1) \tag{7}$$

The dot product and the cross product are used to obtain the cosine and sine of the preceding angles in terms of the observations so that the final form of the triangle constraint equation is now given by

$$\mathbf{w}_2^T A\mathbf{v}_2 = \mathbf{w}_2^T\mathbf{w}_1\mathbf{v}_1^T\mathbf{v}_2 + \|\mathbf{w}_1\times\mathbf{w}_2\|\|\mathbf{v}_1\times\mathbf{v}_2\| \tag{8}$$

From Eq. (8) it is seen that the triangle constraint effectively replaces the information given by the angle observations. Since the two constraints are functions of the observations (not just the parameters) and this approach is a deterministic solution, the constraints can be considered to be observation equations. Therefore the angle observation can be rewritten as

$$d = \mathbf{w}_2^T\mathbf{w}_1\mathbf{v}_1^T\mathbf{v}_2 + \|\mathbf{w}_1\times\mathbf{w}_2\|\|\mathbf{v}_1\times\mathbf{v}_2\| \tag{9}$$

where Eq. (2) has been used.

III. Constrained Solution

Considering the observations shown in Fig. 2, to determine the full attitude between the $\mathcal{B}_1$ and $\mathcal{B}_2$ frames the attitude matrix must satisfy the following equations:

$$\mathbf{w}_1 = A\mathbf{v}_1 \tag{10a}$$

$$\mathbf{w}_2^T\mathbf{w}_1\mathbf{v}_1^T\mathbf{v}_2 + \|\mathbf{w}_1\times\mathbf{w}_2\|\|\mathbf{v}_1\times\mathbf{v}_2\| = d = \mathbf{w}_2^T A\mathbf{v}_2 \tag{10b}$$

Again note, here it is assumed that in an inertial frame the LOS vectors $\mathbf{v}_1$ and $\mathbf{w}_1$ are parallel. Also note that from Fig. 2 no observation information is required from the third object to either $\mathcal{B}_1$ or $\mathcal{B}_2$. Hence, no information such as position is required for this object to determine the relative attitude. A solution for the attitude satisfying Eq. (10) is discussed in [15] and will be utilized to form a solution for the constrained problem discussed here. The solution for the rotation matrix that satisfies Eq. (10) can be found by first finding a rotation matrix that satisfies the first equation and then finding the angle that one must rotate about the reference direction to align the two remaining vectors so that the geometrical constraint is satisfied. The first rotation can be found by rotating about any direction by some angle, where $B = R(\mathbf{n}_1, \theta)$ is a general rotation about some axis rotation, that satisfies Eq. (10a). The choice of the initial rotation axis is arbitrary, here the vector between the two reference direction vectors is used and the rotation is as follows:

$$B = \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{(1 + \mathbf{v}_1^T\mathbf{w}_1)} - I_{3\times 3} \tag{11}$$

where $\mathbf{n}_1 = (\mathbf{w}_1 + \mathbf{v}_1)/\|\mathbf{w}_1 + \mathbf{v}_1\|$, $\theta = \pi$, and $I_{3\times 3}$ is a 3×3 identity matrix. This rotation matrix will align the LOS vectors between frames, but the frames could still have some rotation about this vector, so therefore the angle about this axis must be determined to solve the second equation. To do so the vector $\mathbf{w}^*$ is first defined, which is the vector produced after applying the rotation B on the vector $\mathbf{v}_2$. Then a second rotation needed to map $\mathbf{v}_2$ properly to the $\mathcal{B}_2$ frame with $\mathbf{w}^* = B\mathbf{v}_2$ is used. Since the rotation axis is the $\mathbf{w}_1$ vector, this vector will be invariant under this transformation and the solution to the full attitude can be written as $A = R(\mathbf{n}_2, \theta)B$. The two approaches described in Sec. are used to solve for this rotation angle.

A. Triangle Constraint Solution

Consider solving for the angle rotation by substituting $A = R(\mathbf{n}_2, \theta)B$ into Eq. (10b) and finding an angle θ that satisfies this equation. So it is desired to find a rotation that satisfies $d = \mathbf{w}_2^T R(\mathbf{n}_2, \theta)\mathbf{w}^*$ where

$$R(\mathbf{n}_2, \theta) = I_{3\times 3}\cos(\theta) + (1 - \cos(\theta))\mathbf{n}_2\mathbf{n}_2^T - \sin(\theta)[\mathbf{n}_2\times] \tag{12}$$

Substituting Eq. (12), and with $\mathbf{n}_2 = \mathbf{w}_1$, into Eq. (10b) and using $[\mathbf{w}_1\times]^2 = -I_{3\times 3} + \mathbf{w}_1\mathbf{w}_1^T$ leads to

$$d = \mathbf{w}_2^T(\mathbf{w}_1\mathbf{w}_1^T - \cos(\theta)[\mathbf{w}_1\times]^2 - \sin(\theta)[\mathbf{w}_1\times])\mathbf{w}^* \tag{13}$$

Using $\mathbf{w}^* = B\mathbf{v}_2$, applying the triangle constraint in Eq. (9) and rearranging yields

$$\mathbf{w}_2^T\mathbf{w}_1(\mathbf{w}_1^T B - \mathbf{v}_1^T)\mathbf{v}_2 - \|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\| = \cos(\theta)(\mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*) + \sin(\theta)(\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*) \tag{14}$$

Since the initial rotation B aligns $\mathbf{w}_1$ and $\mathbf{v}_1$, the first term on the left-hand-side of Eq. (14) is zero, so that

$$-1 = \cos(\theta)\frac{\mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*}{\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\|} + \sin(\theta)\frac{\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*}{\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\|} \tag{15}$$

The following identity $\cos(\theta)\cos(\beta) + \sin(\theta)\sin(\beta) = -1$ is now used to solve for θ:

$$\theta = \operatorname{atan2}(\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*, \mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*) + \pi \tag{16}$$

Note that if either $\mathbf{w}_1$ is parallel to $\mathbf{w}_2$ or $\mathbf{v}_1$ is parallel to $\mathbf{v}_2$, then a solution is not possible because the denominators in Eq. (15) are zero. This clearly results in an observable system, which can be seen in Fig. 2 when the vectors are parallel.

B. Planar Constraint Solution

Consider solving for the rotation angle using the planar constraint in Eq. (3), then

$$0 = \mathbf{w}_2^T[\mathbf{w}_1\times]R(\mathbf{n}_2, \theta)\mathbf{w}^* \tag{17}$$

Substituting Eq. (12), and with $\mathbf{n}_2 = \mathbf{w}_1$, into Eq. (17) gives

$$0 = \mathbf{w}_2^T[\mathbf{w}_1\times](I_{3\times3}\cos(\theta) + (1-\cos(\theta))\mathbf{w}_1\mathbf{w}_1^T - \sin(\theta)[\mathbf{w}_1\times])\mathbf{w}^* \tag{18}$$

Expanding out this expression yields

$$(\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*)\cos(\theta) = (\mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*)\sin(\theta) \tag{19}$$

Notice that if Eq. (19) is divided by -1 then the equation would be unchanged but the solution for the angle θ would differ by π. Therefore, using the planar constraint the solution for the angle θ can be written as

$$\theta = \operatorname{atan2}(\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*, \mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*) + \phi \tag{20}$$

where $\phi = 0$ or π. Therefore an ambiguity exists when using this approach but it is important to note that one of the possible solutions for this approach is equivalent to the triangle constraint case.

C. Final Solution

Since the triangle constraint solution has no ambiguity this is the approach that is adopted and the solution for the attitude can be written as $A = R(\mathbf{w}_1, \theta)B$. The solution is now summarized:

$$B = \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{(1 + \mathbf{v}_1^T\mathbf{w}_1)} - I_{3\times3} \tag{21a}$$

$$\mathbf{w}^* = B\mathbf{v}_2 \tag{21b}$$

$$\theta = \operatorname{atan2}(\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*, \mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*) + \pi \tag{21c}$$

$$R(\mathbf{w}_1, \theta) = \cos(\theta)I_{3\times3} + (1-\cos(\theta))\mathbf{w}_1\mathbf{w}_1^T - \sin(\theta)[\mathbf{w}_1\times] \tag{21d}$$

$$A = R(\mathbf{w}_1, \theta)B \tag{21e}$$

This result shows that for any formation of two vehicles a deterministic solution will exist using one direction and one angle. Because of the fact that the case shown here is truly deterministic there is no need to minimize a cost function. It is very important to note that without the resolution of the attitude ambiguity any covariance development might not have any meaning since although the covariance might take a small value if the wrong possible attitude is used then the error might be fairly large and not bounded by the attitude covariance.

The solution in Eq. (21) can be rewritten without the use of any transcendental functions. From Eq. (15) the following relationships can be seen:

$$\cos(\theta) = -\frac{\mathbf{w}_2^T[\mathbf{w}_1\times]^2\mathbf{w}^*}{\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\|} \tag{22a}$$

$$\sin(\theta) = -\frac{\mathbf{w}_2^T[\mathbf{w}_1\times]\mathbf{w}^*}{\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\|} \tag{22b}$$

Substituting Eqs. (21a) and (21b) into Eq. (22) leads to $\cos(\theta) = -b/c$ and $\sin(\theta) = -a/c$ with

$$a = \mathbf{w}_2^T[\mathbf{w}_1\times]([\mathbf{w}_1\times] + [\mathbf{v}_1\times])[\mathbf{v}_1\times]\mathbf{v}_2 \tag{23a}$$

$$b = \mathbf{w}_2^T[\mathbf{w}_1\times]([\mathbf{w}_1\times][\mathbf{v}_1\times] - I_{3\times3})[\mathbf{v}_1\times]\mathbf{v}_2 \tag{23b}$$

$$c = (1 + \mathbf{v}_1^T\mathbf{w}_1)\|\mathbf{w}_1 \times \mathbf{w}_2\|\|\mathbf{v}_1 \times \mathbf{v}_2\| \tag{23c}$$

Note that $c = \sqrt{a^2 + b^2}$. Then the matrix R in Eq. (21d) is given by

$$R = -\frac{b}{c}I_{3\times3} + \left(1 + \frac{b}{c}\right)\mathbf{w}_1\mathbf{w}_1^T + \frac{a}{c}[\mathbf{w}_1\times] \tag{24}$$

Noting that $\mathbf{w}_1\mathbf{w}_1^T B = \mathbf{w}_1\mathbf{v}_1^T$ then the solution in Eq. (21e) can be rewritten as

$$A = \frac{b}{c}\left(I_{3\times3} - \frac{(\mathbf{w}_1 + \mathbf{v}_1)(\mathbf{w}_1 + \mathbf{v}_1)^T}{(1 + \mathbf{v}_1^T\mathbf{w}_1)} + \mathbf{w}_1\mathbf{v}_1^T\right) + \frac{a}{c}[\mathbf{w}_1\times]\left(\frac{\mathbf{v}_1\mathbf{w}_1^T + \mathbf{v}_1\mathbf{v}_1^T}{(1 + \mathbf{v}_1^T\mathbf{w}_1)} - I_{3\times3}\right) + \mathbf{w}_1\mathbf{v}_1^T \tag{25}$$

Note in practice the measured quantities, which are discussed in the next section, are used in place of the observed quantities shown in Eq. (21), and Eqs. (23) and (25).

IV. Sensor Model

LOS observations between multiple vehicles can be obtained using standard light-beam and focal plane-detector technology. One such system is the VISNAV system [18], which consists of a position sensing diode as the focal plane that captures incident light from a beacon omitted from a neighboring vehicle from which a LOS vector can be determined. The light source is such that the system can achieve selective vision. This sensor has the advantage of having a small size and a very wide field-of-view (FOV). Another system for obtaining LOS information between vehicles can be based on laser communication hardware [19]. The use of laser communication devices has increased in recent years and the accuracy of the LOS information obtained from these devices is comparable to the VISNAV system. LOS observations to the common object can be obtained through standard camera-based tracking technology. All of the aforementioned LOS observations can be modeled using the sensor model shown in this section.

The measurement can be expressed as coordinates in the focal plane, denoted by α and β. The focal plane coordinates can be written in a 2×1 vector $\mathbf{m} \equiv [\alpha \quad \beta]^T$ and the measurement model follows

$$\tilde{\mathbf{m}} = \mathbf{m} + \mathbf{w}_m \tag{26}$$

A typical noise model used to describe the uncertainty, $\mathbf{w}_m$, in the focal plane coordinate measurements is given as

$$\mathbf{w}_m \sim \mathcal{N}(\mathbf{0}, R^{\text{FOCAL}}) \tag{27a}$$

$$R^{\text{FOCAL}} = \frac{\sigma^2}{1 + d(\alpha^2 + \beta^2)} \begin{bmatrix} (1 + d\alpha^2)^2 & (d\alpha\beta)^2 \\ (d\alpha\beta)^2 & (1 + d\beta^2)^2 \end{bmatrix} \quad (27b)$$

where σ^2 is the variance of the measurement errors associated with α and β, and d is on the order of 1. The covariance for the focal plane measurements is a function of the true values and this covariance realistically increases as the distance from the boresight increases. The measurement error associated with the focal plane measurements results in error in the measured LOS vector. A general sensor LOS observation can be expressed in unit vector form given by

$$\mathbf{b} = \frac{1}{\sqrt{f + \alpha^2 + \beta^2}} \begin{bmatrix} \alpha \\ \beta \\ f \end{bmatrix} \quad (28)$$

where f denotes the focal length. The LOS observation has two independent parameters α and β. Therefore in the presence of random noise in these parameters the LOS vector still must maintain a unit norm. Although the LOS measurement noise must lie on the unit sphere the measurement noise can be approximated as additive noise, given by

$$\tilde{\mathbf{b}} = \mathbf{b} + \boldsymbol{\upsilon} \quad (29)$$

with

$$\boldsymbol{\upsilon} \sim \mathcal{N}(\mathbf{0}, \Omega) \quad (30)$$

where $\boldsymbol{\upsilon}$ is assumed to be a Gaussian random vector with zero mean and covariance Ω. Reference [13] has shown that the probability density for unit vector measurements lies on a sphere and can accurately be approximated by a density on a plane tangent to the vector for a small FOV sensors. This approximation is known as the QUEST measurement model [13], which characterizes the LOS noise process resulting from the focal plane model as

$$\Omega \equiv E\{\boldsymbol{\upsilon}\boldsymbol{\upsilon}^T\} = \sigma^2(I_{3\times3} - \mathbf{b}\mathbf{b}^T) \quad (31)$$

It is clear that this is only valid for a small FOV in which a tangent plane closely approximates the surface of a unit sphere. For wide-FOV sensors, a more accurate measurement covariance is shown in [20]. This formulation employs a first-order Taylor series approximation about the focal plane axes. The partial derivative operator is used to linearly expand the focal plane covariance in Eq. (27), given by (for $f = 1$)

$$J = \frac{\partial \mathbf{b}}{\partial \mathbf{m}} = \frac{1}{\sqrt{1 + \alpha^2 + \beta^2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} - \frac{1}{1 + \alpha^2 + \beta^2} \mathbf{b}\mathbf{m}^T \quad (32)$$

Then the wide-FOV covariance model is given by

$$\Omega = J R^{\text{FOCAL}} J^T \quad (33)$$

If a small FOV model is valid, then Eq. (33) can still be used, but is nearly identical to Eq. (31). For both equations, Ω is a 3×3 covariance matrix for a unit vector measurement with two independent parameters and therefore must be singular. A nonsingular covariance matrix for the LOS measurements can be obtained by a rank-one update to Ω:

$$\Omega_{\text{new}} = \Omega + \frac{1}{2}\text{trace}(\Omega)\mathbf{b}\mathbf{b}^T \quad (34)$$

which can be used without loss in generality to develop attitude error covariance expressions [16]. Equation (33) represents the covariance for the LOS measurements in their respective body frame shown in Fig. 2. Replacing $\mathbf{b}$ with respective true vectors and $\tilde{\mathbf{b}}$ with respective measured vectors, the four measurement models are summarized by

$$\tilde{\mathbf{w}}_1 = \mathbf{w}_1 + \upsilon_{w1}, \qquad \upsilon_{w1} \sim \mathcal{N}(\mathbf{0}, R_{w_1}) \quad (35a)$$

$$\tilde{\mathbf{w}}_2 = \mathbf{w}_2 + \upsilon_{w2}, \qquad \upsilon_{w2} \sim \mathcal{N}(\mathbf{0}, R_{w_2}) \quad (35b)$$

$$\tilde{\mathbf{v}}_1 = \mathbf{v}_1 + \upsilon_{v1}, \qquad \upsilon_{v1} \sim \mathcal{N}(\mathbf{0}, R_{v_1}) \quad (35c)$$

$$\tilde{\mathbf{v}}_2 = \mathbf{v}_2 + \upsilon_{v2}, \qquad \upsilon_{v2} \sim \mathcal{N}(\mathbf{0}, R_{v_2}) \quad (35d)$$

Since in practice each vehicle will have their own set of LOS measurement devices, then the measurements in Eq. (35) can be assumed to be uncorrelated. This assumption will be used in the attitude covariance derivation.

V. Attitude Error Covariance Matrix

The covariance matrix represents the local second-order moment of the probability density function (PDF) of the error in the estimate, and barring the case where there are two solutions in the same vicinity it is independent of the solution chosen. The covariance matrix for an attitude estimate is defined as the covariance of a small angle rotation taking the true attitude to the estimated attitude. Typically the small Euler angles are used to parameterize the attitude matrix. For the derivation of the covariance the two sets of measurement equations that satisfy the final solution in Sec. III.C are considered, but only one set results in an accurate covariance expression. It is important to note that although a closed-form solution exists for the attitude, the covariance is derived using approximate linear methods because the measurement model is nonlinear in both the attitude and measurement noise. The attitude linearization has been shown to be valid for the case of small attitude errors [21], which is also assumed here. The noise linearization will produce an accurate covariance expression for large signal-to-noise ratios; even with errors of 1° this is not a concern [21].

A. Planar Constraint Equation Covariance

The first set that is considered incorporates the planar constraint representation in the scalar measurement equation. This set will be considered first because it results in an accurate expression for the attitude error covariance. Using measurement and estimate notation specifically, Eq. (10) is now written as

$$\tilde{\mathbf{w}}_1 = \hat{A}\tilde{\mathbf{v}}_1 \quad (36a)$$

$$0 = \tilde{\mathbf{w}}_2^T[\tilde{\mathbf{w}}_1 \times]\hat{A}\tilde{\mathbf{v}}_2 \quad (36b)$$

where $\hat{A}$ is the estimated (determined) attitude matrix. The preceding measurement equations are indeed satisfied by the constrained solution but these equations have a twofold ambiguity in the rotation angle θ at $\theta + \pi$. However, a covariance derived for these equations will be valid for both cases. Therefore the preceding equations can be used to derive a covariance for the constrained solution presented in Sec. III.C. Equation (36) can be rewritten in vector form as

$$\tilde{\mathbf{y}} = \mathbf{h}(A_{\text{true}}) + \boldsymbol{\Delta} \quad (37)$$

where $\mathbf{h}(A_{\text{true}}) = [\,(A_{\text{true}}\mathbf{v}_1)^T \quad \mathbf{w}_2^T[\mathbf{w}_1\times]A_{\text{true}}\mathbf{v}_2\,]^T$ is the true output which is a function of the true attitude A_{true}. Also $\tilde{\mathbf{y}}$ is the measurement vector defined by $\tilde{\mathbf{y}} = [\,\tilde{\mathbf{w}}_1^T \quad 0\,]^T$. The other quantities are taken to be deterministic but their errors are considered in $\boldsymbol{\Delta}$ which is a random noise vector describing the error in the measurement vector $\tilde{\mathbf{y}}$. The PDF of $\boldsymbol{\Delta}$ is assumed to be Gaussian with zero mean, i.e. $E\{\boldsymbol{\Delta}\} = 0$, so that

$$\boldsymbol{\Delta} \sim \mathcal{N}(\mathbf{0}, \mathcal{R}) \quad (38)$$

The quantity $\boldsymbol{\Delta} = [\,\boldsymbol{\Delta}_1^T \quad \Delta_2\,]^T$, where $\boldsymbol{\Delta}_1$ and Δ_2 are the errors in the first and second measurements, respectively. Then the covariance of $\boldsymbol{\Delta}$ is defined as

$$\mathcal{R} = \begin{bmatrix} R_{\Delta_1} & R_{\Delta_1\Delta_2} \\ R^T_{\Delta_1\Delta_2} & R_{\Delta_2} \end{bmatrix} \tag{39}$$

where $R_{\Delta_1} = E\{\boldsymbol{\Delta}_1\boldsymbol{\Delta}_1^T\}$, $R_{\Delta_2} = E\{\Delta_2^2\}$ and $R_{\Delta_1\Delta_2} = E\{\boldsymbol{\Delta}_1\Delta_2\}$. To determine the covariance of the measurement vector $\tilde{\mathbf{y}}$, the covariance of $\boldsymbol{\Delta}_1$ and variance of Δ_2, along with their cross correlation term must be determined. This can be done by calculating $\tilde{\mathbf{y}} - \mathbf{h}(A_{\text{true}})$ for the LOS and scalar measurement equations.

1. *Covariance for the LOS Measurement Equation*

The error vector component due to the LOS measurement can be written as

$$\boldsymbol{\Delta}_1 = \tilde{\mathbf{w}}_1 - A_{\text{true}}\tilde{\mathbf{v}}_1 \tag{40}$$

Substituting Eqs. (35a) and (35c) gives

$$\boldsymbol{\Delta}_1 = \mathbf{w}_1 - A_{\text{true}}\mathbf{v}_1 + \boldsymbol{v}_{w1} - A_{\text{true}}\boldsymbol{v}_{v1} \tag{41}$$

Equation (41) is a linear addition of two Gaussian noise terms and therefore $\boldsymbol{\Delta}_1$ is also Gaussian. Considering Eq. (1), Eq. (41) becomes

$$\boldsymbol{\Delta}_1 = \boldsymbol{v}_{w1} - A_{\text{true}}\boldsymbol{v}_{v1} \tag{42}$$

Then taking the expectation of $\boldsymbol{\Delta}_1\boldsymbol{\Delta}_1^T$ gives the following covariance expression for $\boldsymbol{\Delta}_1$:

$$R_{\Delta_1} = R_{w_1} + A_{\text{true}}R_{v_1}A^T_{\text{true}} \tag{43}$$

Here the measurement vector $\mathbf{v}_1$ as well as the vector vector $\mathbf{w}_1$ have uncertainty and therefore the covariance of the LOS measurement is a function of both their noise characteristics. Since these covariance matrices are represented with respect to two different body frames, then R_{v_1} must be rotated into the $\mathcal{B}_2$ frame, therefore R_{Δ_1} is a function of the true attitude. The true attitude is unknown in practice but the true attitude can be effectively replaced by the estimated attitude in the covariance equation with only second-order error effects [21].

2. *Covariance for the Scalar Measurement Equation*

The expression for Δ_2 can be found by

$$\Delta_2 = 0 - \tilde{\mathbf{w}}_2^T[\mathbf{w}_1\times]A_{\text{true}}\tilde{\mathbf{v}}_2 \tag{44}$$

Substituting Eqs. (35a), (35b), and (35d) into Eq. (44) gives

$$\Delta_2 = -(\mathbf{w}_2 + \boldsymbol{v}_{w2})^T[(\mathbf{w}_1 + \boldsymbol{v}_{w1})\times]A_{\text{true}}(\mathbf{v}_2 + \boldsymbol{v}_{v2}) \tag{45}$$

Since all measurement noise terms can be assumed to be small, then second-order terms can be neglected in the measurement noise terms and the expression for Δ_2 becomes

$$\Delta_2 = \mathbf{w}_2^T[A_{\text{true}}\mathbf{v}_2\times]\boldsymbol{v}_{w1} - \mathbf{w}_2^T[\mathbf{w}_1\times]A_{\text{true}}\boldsymbol{v}_{v2} + (A_{\text{true}}\mathbf{v}_2)^T[\mathbf{w}_1\times]\boldsymbol{v}_{w2} \tag{46}$$

Taking the expectation of $\Delta_2\Delta_2^T$ and neglecting cross correlation terms, since the measurement errors are assumed to be uncorrelated, then the covariance expression for Δ_2 is

$$\begin{aligned} R_{\Delta_2} = -\Big\{&\mathbf{w}_2^T[A_{\text{true}}\mathbf{v}_2\times]R_{w_1}[A_{\text{true}}\mathbf{v}_2\times]\mathbf{w}_2 \\ &+ (A_{\text{true}}\mathbf{v}_2)^T[\mathbf{w}_1\times]R_{w_2}[\mathbf{w}_1\times](A_{\text{true}}\mathbf{v}_2) \\ &+ \mathbf{w}_2^T[\mathbf{w}_1\times]A_{\text{true}}R_{v_2}A^T_{\text{true}}[\mathbf{w}_1\times]\mathbf{w}_2\Big\} \end{aligned} \tag{47}$$

3. *Cross Correlation*

The off-diagonal term of the $\mathcal{R}$ matrix represents the cross correlations between the first and second measurement equations. This term can be found by taking the expectation of $\boldsymbol{\Delta}_1\Delta_2$. This expectation can be written as

$$\begin{aligned} R_{\Delta_1\Delta_2} = E\{&(\boldsymbol{v}_{w1} - A_{\text{true}}\boldsymbol{v}_{v1})(\mathbf{w}_2^T[A_{\text{true}}\mathbf{v}_2\times]\boldsymbol{v}_{w1} \\ &- \mathbf{w}_2^T[\mathbf{w}_1\times]A_{\text{true}}\boldsymbol{v}_{v2} + (A_{\text{true}}\mathbf{v}_2)^T[\mathbf{w}_1\times]\boldsymbol{v}_{w2})\} \end{aligned} \tag{48}$$

Since it is assumed that the measurements are uncorrelated, after expanding out this expectation and evaluating it, the correlation between the first and second measurement equations can be written as

$$R_{\Delta_1\Delta_2} = -R_{w_1}[A_{\text{true}}\mathbf{v}_2\times]\mathbf{w}_2 \tag{49}$$

The cross correlation term only involves the error characteristics of $\mathbf{w}_1$ since this is the only measurement that is common in both measurement equations.

4. *Linearized Covariance Expression*

An approximated covariance expression for estimated attitude error can be obtained by linearizing the attitude matrix utilizing a small Euler rotation vector representation, where the error model given by Eq. (37) is used. The attitude matrix can be parameterized by the error-angle vector taking the true attitude to the estimated given by $\hat{A} = e^{-[\boldsymbol{\delta\alpha}\times]}A_{\text{true}} \approx (I_{3\times3} - [\boldsymbol{\delta\alpha}\times])A_{\text{true}}$ where $\boldsymbol{\delta\alpha}$ represents the small roll, pitch and yaw error rotations. The attitude covariance is defined as

$$P_{\boldsymbol{\delta\alpha\delta\alpha}} = E\{\boldsymbol{\delta\alpha}\boldsymbol{\delta\alpha}^T\} \tag{50}$$

From Eq. (36) the following expression is given:

$$\tilde{\mathbf{y}} = \mathbf{h}(\hat{A}) \tag{51}$$

Substituting the expression for the first-order expansion of the estimated attitude into $\mathbf{h}(\hat{A})$ gives

$$\mathbf{h}(\hat{A}) = \begin{bmatrix} A_{\text{true}}\mathbf{v}_1 - [A_{\text{true}}\mathbf{v}_1\times]\boldsymbol{\delta\alpha} \\ \mathbf{w}_2^T[\mathbf{w}_1\times]A_{\text{true}}\mathbf{v}_2 - \mathbf{w}_2^T[\mathbf{w}_1\times][A_{\text{true}}\mathbf{v}_2\times]\boldsymbol{\delta\alpha} \end{bmatrix} \tag{52}$$

which can be written as $\mathbf{h}(\hat{A}) = \mathbf{h}(A_{\text{true}}) + H\boldsymbol{\delta\alpha}$, where $H = [[A_{\text{true}}\mathbf{v}_1\times] \quad - \quad [A_{\text{true}}\mathbf{v}_2\times][\mathbf{w}_1\times]\mathbf{w}_2]^T$. Substituting this expression into Eq. (51) and subtracting $\mathbf{h}(A_{\text{true}})$ from both sides of the equation leads to

$$\tilde{\mathbf{y}} - \mathbf{h}(A_{\text{true}}) = H\boldsymbol{\delta\alpha} \tag{53}$$

This is equivalent to $\boldsymbol{\Delta} = H\boldsymbol{\delta\alpha}$, so a least-squares type solution can be formed for the error rotation angle since the equation is now linear. The least-squares solution for $\boldsymbol{\delta\alpha}$ can be written as

$$\boldsymbol{\delta\alpha} = (H^T\mathcal{R}^{-1}H)^{-1}H^T\mathcal{R}^{-1}\boldsymbol{\Delta} \tag{54}$$

This equation can be used in a nonlinear least-squares solution for the attitude by using $\boldsymbol{\delta\alpha}$ as the estimated correction, but in this case it is more useful for obtaining an expression for the attitude covariance, which is simply computed by

$$P_{\boldsymbol{\delta\alpha\delta\alpha}} = (H^T\mathcal{R}^{-1}H)^{-1} = \left(\begin{bmatrix} -[A_{\text{true}}\mathbf{v}_1\times] \\ -\mathbf{w}_2^T[\mathbf{w}_1\times][A_{\text{true}}\mathbf{v}_2\times] \end{bmatrix}^T \begin{bmatrix} R_{\Delta_1} & R_{\Delta_1\Delta_2} \\ R^T_{\Delta_1\Delta_2} & R_{\Delta_2} \end{bmatrix}^{-1} \begin{bmatrix} -[A_{\text{true}}\mathbf{v}_1\times] \\ -\mathbf{w}_2^T[\mathbf{w}_1\times][A_{\text{true}}\mathbf{v}_2\times] \end{bmatrix}\right)^{-1} \tag{55}$$

Once again the true attitude can effectively be replaced with the estimated attitude.

B. Sensitivity for the Triangle Constraint Measurement Equation

The covariance for the attitude solution was derived using the planar constraint measurement equations but the algorithm in Sec. III.C used the triangle constraint to forgo the solution ambiguity. The two solutions have been shown to be equivalent and therefore the covariance expression should be equivalent. Deriving the covariance using the triangle constraint equation is now considered. Similarly the linearized error model is derived to determine the estimate attitude covariance. As in the case shown previously, the expression for the attitude covariance is $P_{\delta\alpha\delta\alpha} = (H^T \mathcal{R}^{-1} H)^{-1}$, but new definitions for H and $\mathcal{R}$ are now required since the triangle constraint measurement equation is different than the planar constraint measurement equation. The measurement equations can be put into the form of Eq. (51) where $\tilde{\mathbf{y}} = [\mathbf{w}_1^T \quad \mathbf{w}_2^T \mathbf{w}_1 \mathbf{v}_1^T \mathbf{v}_2 + \|\mathbf{w}_1 \times \mathbf{w}_2\| \|\mathbf{v}_1 \times \mathbf{v}_2\|]^T$ and $\mathbf{h}(\hat{A}) = [(\hat{A}\mathbf{v}_1)^T \quad \mathbf{w}_2^T \hat{A} \mathbf{v}_2]^T$. Substituting for $\hat{A}$ gives

$$\mathbf{h}(\hat{A}) = \begin{bmatrix} A_{\text{true}} \mathbf{v}_1 - [A_{\text{true}} \mathbf{w}_1 \times] \boldsymbol{\delta\alpha} \\ \mathbf{w}_2^T A_{\text{true}} \mathbf{v}_2 - \mathbf{w}_2^T [A_{\text{true}} \mathbf{v}_2 \times] \boldsymbol{\delta\alpha} \end{bmatrix} \tag{56}$$

Then the sensitivity matrix can be written as

$$H = \begin{bmatrix} -[A_{\text{true}} \mathbf{v}_1 \times] \\ -\mathbf{w}_2^T [A_{\text{true}} \mathbf{v}_2 \times] \end{bmatrix} \tag{57}$$

The covariance $P_{\delta\alpha\delta\alpha}$ exists if and only if H has full rank. Therefore for the covariance to exist H must have rank 3, but note that the null space of $\mathbf{w}_2^T [A_{\text{true}} \mathbf{v}_2 \times]$ is spanned by any vector on the plane formed by $A_{\text{true}} \mathbf{v}_2$ and $\mathbf{w}_2$. Therefore the $\mathbf{w}_1$ which lies on this plane is in the null space of $\mathbf{w}_2^T [A_{\text{true}} \mathbf{v}_2 \times]$. Since the $\mathbf{w}_1$ vector forms the null vector of $[A_{\text{true}} \mathbf{v}_1 \times]$ the H matrix is rank deficient and the covariance using the linearized error model for this set of measurement equations does not exist.

To determine why there is a difference between the two approaches of representing the problem, an illustration by example is shown. Consider the following configuration:

$$\mathbf{w}_1 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \tag{58}$$

The $\mathbf{w}_1$ vector describes the LOS observation and as mentioned previously it is known that the rotation about this axis is unobservable and $\mathbf{w}_3 = A_{\text{true}} \mathbf{v}_2$. Consider the following attitude matrix which is a rotation about the $\mathbf{w}_1$ vector, which is the unobservable direction:

$$A(\mathbf{w}_1, \theta) = I_{3\times3} \cos(\theta) + (1 - \cos(\theta)) \mathbf{w}_1 \mathbf{w}_1^T - \sin(\theta) [\mathbf{w}_1 \times] \tag{59}$$

Then consider the triangle constraint equation under this rotation:

$$0 = d = \mathbf{w}_2^T (I_{3\times3} \cos(\theta) + (1 - \cos(\theta)) \mathbf{w}_1 \mathbf{w}_1^T - \sin(\theta) [\mathbf{w}_1 \times]) \mathbf{w}_3 \tag{60}$$

Expanding out this expression results in

$$0 = d = \mathbf{w}_2^T \mathbf{w}_3 \cos(\theta) + (1 - \cos(\theta)) \mathbf{w}_2^T \mathbf{w}_1 \mathbf{w}_1^T \mathbf{w}_3 - \sin(\theta) \mathbf{w}_2^T [\mathbf{w}_1 \times] \mathbf{w}_3 \tag{61}$$

Given that the triangle configuration lies on the y-z plane, then $\mathbf{w}_2^T [\mathbf{w}_1 \times] \mathbf{w}_3 = 0$ and $\mathbf{w}_2^T \mathbf{w}_3 = 0$ since these two vectors are orthogonal. It can be shown that for this configuration $\mathbf{w}_2^T \mathbf{w}_1 \mathbf{w}_1^T \mathbf{w}_3 = \frac{1}{2}$. Then Eq. (61) simplifies to

$$0 = d = \frac{1}{2}(1 - \cos(\theta)) \tag{62}$$

The planar constraint equation for this configuration is now considered, which can be written as

$$0 = \mathbf{w}_2^T [\mathbf{w}_1 \times] (I_{3\times3} \cos(\theta) + (1 - \cos(\theta)) \mathbf{w}_1 \mathbf{w}_1^T - \sin(\theta) [\mathbf{w}_1 \times]) \mathbf{w}_3 \tag{63}$$

Expanding out this expression results in

$$0 = \cos(\theta) \mathbf{w}_2^T [\mathbf{w}_1 \times] \mathbf{w}_3 + (1 - \cos(\theta)) \mathbf{w}_2^T [\mathbf{w}_1 \times] \mathbf{w}_1 \mathbf{w}_1^T \mathbf{w}_3 - \sin(\theta) \mathbf{w}_2^T [\mathbf{w}_1 \times][\mathbf{w}_1 \times] \mathbf{w}_3 \tag{64}$$

Again using the fact that the triangle configuration lies on the y-z plane, then $\mathbf{w}_2^T [\mathbf{w}_1 \times] \mathbf{w}_3 = 0$ and $[\mathbf{w}_1 \times] \mathbf{w}_1$ is equivalently zero. It can be shown that for this configuration $\mathbf{w}_2^T [\mathbf{w}_1 \times][\mathbf{w}_1 \times] \mathbf{w}_3 = -\frac{1}{2}$. Then Eq. (64) simplifies to

$$0 = \frac{1}{2} \sin(\theta) \tag{65}$$

The results in Eqs. (62) and (65) are useful to examine the sensitivity in these equations for rotation about the $\mathbf{w}_1$ vector, which is unobservable with just the $\mathbf{w}_1$ LOS observation. The scalar equations should provide sensitivity in this direction to ensure that H is full rank. To arrive at the linearized noise model a small angle approximation in Eqs. (62) and (65) has been used. Under the small angle approximation $\cos(\theta) \approx 1$ and $\sin(\theta) \approx \theta$, the sensitivity of $\cos(\theta)$ with respect to θ is zero while the sensitivity of $\sin(\theta)$ is one. This results in no sensitivity in the rotation around the vector direction for the triangle constraint set of equations, making the H matrix rank deficient. Therefore the planar constraint is the only approach which gives a reasonable covariance for the linearized error models. So one can think of $d = \mathbf{w}_2^T A_{\text{true}} \mathbf{v}_2$ as the "cosine" constraint because it employs the dot product and $\mathbf{w}_2^T [\mathbf{w}_1 \times] A_{\text{true}} \mathbf{v}_2$ as the "sine" constraint because it employs the cross product. The two differ in that a linearization is performed about a different point resulting in the difference between their resulting covariance expressions. The expression in Eq. (55) is used to show that the derived attitude-error covariance does indeed bound these errors in a 3σ sense.

VI. Sensitivity to Out-of-Plane Deflection

In this section an expression for the sensitivity of the attitude error is derived for the case that a perfect triangle configuration is not given. The sensitivity is important because any deviations from the triangle constraint will cause attitude errors in the solution. The sensitivity analysis shown here can help an analyst study out-of-plane deflections and investigate their effects on the attitude solution. The triangle assumption can only be violated by the case where one of the observation vectors is out of the plane containing the other two observations. Since the $\mathbf{w}_1$ and $\mathbf{v}_1$ vectors are common LOS observations expressed in different coordinates, then by definition they must be in the same direction and therefore these vectors cannot be out of the plane. The $\mathbf{v}_2$ and $\mathbf{w}_2$ vectors are the only two vectors that can be out-of-plane. Since one of these vectors has to be used with the $\mathbf{w}_1$ and $\mathbf{v}_1$ direction to define a plane, then only one observation vector needs to be chosen to be out of the plane.

Consider rotating the $\mathbf{v}_2$ vector out of the plane by an angle Φ. Then the resulting out-of-plane vector can be defined as

$$\mathbf{v}_\Phi = R(\Phi, \mathbf{e}) \mathbf{v}_2 \tag{66}$$

where $\mathbf{e}$ is the axis of rotation; see Fig. 3. Then it follows that

$$\mathbf{e} = -\frac{[\mathbf{v}_2 \times]^2 \mathbf{v}_1}{\|\mathbf{v}_2 \times \mathbf{v}_1\|} \tag{67}$$

Note that $\|[\mathbf{v}_2 \times]^2 \mathbf{v}_1\| = \|\mathbf{v}_2 \times \mathbf{v}_1\|$. The out-of-plane vector can be written using the definition of the attitude matrix [22]:

$$\mathbf{v}_\Phi = [I_{3\times3} \cos(\Phi) + (1 - \cos(\Phi)) \mathbf{e}\mathbf{e}^T + \sin(\Phi) [\mathbf{e} \times]] \mathbf{v}_2 \tag{68}$$

Noting that $\mathbf{e}^T \mathbf{v}_2 = 0$ simplifies Eq. (68) to give

$$\mathbf{v}_\Phi = (I_{3\times3} \cos(\Phi) + \sin(\Phi) [\mathbf{v}_1 \times] / \|\mathbf{v}_2 \times \mathbf{v}_1\|) \mathbf{v}_2 \tag{69}$$

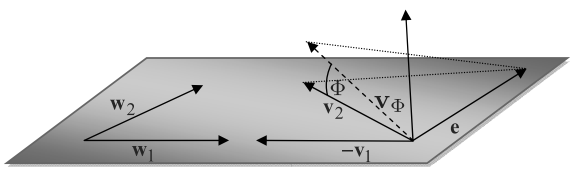

Fig. 3 Out-of-plane geometry.

The goal is to obtain an expression that relates the increase in the attitude error due to the out-of-plane deflection. This can be accomplished using the solution in Eq. (21e) to obtain an expression for the sensitivity of the error to the out-of-plane deflection. The attitude error matrix can be written as

$$\delta A_\Phi = A_\Phi A^T \tag{70}$$

where the matrix A is the attitude matrix formed using the observation set $\{\mathbf{w}_1, \mathbf{v}_1, \mathbf{w}_2, \mathbf{v}_2\}$ and the matrix A_Φ is the attitude matrix formed using the observation set $\{\mathbf{w}_1, \mathbf{v}_1, \mathbf{w}_2, \mathbf{v}_\Phi\}$. As explained previously the only out-of-plane vector is given by rotating $\mathbf{v}_2$ onto $\mathbf{v}_\Phi$. The attitude solution for A is accomplished by two successive rotations, the first rotation is given by the matrix B and the second rotation is given by the matrix R, where $R \equiv R(\mathbf{w}_1, \theta)$ is used for convenience. The solution for the estimated attitude is written as $A = RB$. The matrix B aligns the $\mathbf{v}_1$ and $\mathbf{w}_1$ directions and therefore this matrix is independent of $\mathbf{v}_\Phi$. The second rotation is a simple Euler axis/angle rotation about the $\mathbf{w}_1$ vector by the angle of θ. Its equivalent quaternion [22] is given by

$$\mathbf{q}_R = \begin{bmatrix} \sin(\theta/2)\mathbf{w}_1 \\ \cos(\theta/2) \end{bmatrix} \tag{71}$$

This shows that all of the out-of-error is expressed solely by θ, as seen by Eq. (21c). The attitude matrix associated with $\mathbf{q}_R$ is denoted by R. The quantity θ_Φ is used to represent the error in θ due to the out-of-plane deflection. The quaternion associated with θ_Φ is given by

$$\mathbf{q}_{R_\Phi} = \begin{bmatrix} \sin(\theta_\Phi/2)\mathbf{w}_1 \\ \cos(\theta_\Phi/2) \end{bmatrix} \tag{72}$$

and its associated attitude matrix is R_Φ. Then $A_\Phi = R_\Phi B$, so that

$$\delta A_\Phi = R_\Phi B (RB)^T = R_\Phi B B^T R^T = R_\Phi R^T \tag{73}$$

So the error is only a function of R_Φ and R. Using quaternion multiplication [22] Eq. (73) can be rewritten as

$$\delta \mathbf{q}_{R_\Phi} = \mathbf{q}_{R_\Phi} \otimes \mathbf{q}_R^{-1} \tag{74}$$

where the error quaternion can be related to small angle errors by $\delta\mathbf{q}_{R_\Phi} = [\,\delta\boldsymbol{\alpha}_\Phi^T/2 \quad 1\,]^T$. By carrying out the quaternion multiplication the vector component of the error quaternion can be shown to be given by

$$\delta\boldsymbol{\alpha}_\Phi = 2[\sin(\theta_\Phi/2)\cos(\theta/2) - \cos(\theta_\Phi/2)\sin(\theta/2)]\mathbf{w}_1 \tag{75}$$

By noting that $\sin((\theta_\Phi - \theta)/2) = \sin(\theta_\Phi/2)\cos(\theta/2) - \cos(\theta_\Phi/2)\sin(\theta/2)$ and assuming $(\theta_\Phi - \theta)$ is small, Eq. (75) becomes

$$\delta\boldsymbol{\alpha}_\Phi = (\theta_\Phi - \theta)\mathbf{w}_1 \tag{76}$$

Note that $\mathbf{w}_1$ is independent of Φ and only θ_Φ depends on Φ, where $\mathbf{w}_1$ defines the direction of $\delta\boldsymbol{\alpha}_\Phi$. The magnitude of the small angle vector gives the angle of rotation about $\mathbf{w}_1$, taking A_Φ to A. Calculating the derivative of the magnitude of $\delta\boldsymbol{\alpha}_\Phi$ with respect to Φ quantifies the sensitivity of the solution to the out-of-plane deflection angle. Since $\mathbf{w}_1$ is assumed to be a unit vector the magnitude of the small error angle can be written as $\Theta = \delta\boldsymbol{\alpha}_\Phi^T\mathbf{w}_1$. To consider the sensitivity of the solution to out-of-plane deflection the sensitivity in Θ is considered. Hence the following defined quantity is used to study the sensitivity is $\Theta \equiv (\theta_\Phi - \theta)$. The sensitivity of the solution to out-of-plane deflection is given by

$$\left.\frac{\mathrm{d}\Theta}{\mathrm{d}\Phi}\right|_{\Phi=0} = \left.\frac{\mathrm{d}\theta_\Phi}{\mathrm{d}\mathbf{v}_\Phi^T}\frac{\mathrm{d}\mathbf{v}_\Phi}{\mathrm{d}\Phi}\right|_{\Phi=0} \tag{77}$$

Expressions for the derivatives in Eq. (77) are needed. Using $\mathbf{w}^* = B\mathbf{v}_2$, θ can be written as

$$\theta = \mathrm{atan2}(\mathbf{w}_2^T[\mathbf{w}_1\times]B\mathbf{v}_2, \mathbf{w}_2^T[\mathbf{w}_1\times]^2 B\mathbf{v}_2) + \pi \tag{78}$$

To simplify the derivation of the sensitivity expression Eq. (78) can be rearranged using $\mathbf{w}_1 = B\mathbf{v}_1$ and $\mathbf{v}_2^* = B^T\mathbf{w}_2$. Then by defining $\mathcal{Y} \equiv \mathbf{v}_2^{*^T}[\mathbf{v}_1\times]\mathbf{v}_2$ and $\mathcal{X} \equiv \mathbf{v}_2^{*^T}[\mathbf{v}_1\times]^2\mathbf{v}_2$ the angle θ is given by

$$\theta \equiv \mathrm{atan2}(\mathcal{Y}, \mathcal{X}) + \pi \tag{79}$$

To compute the sensitivity in Eq. (77) first θ_Φ is calculated using $\{\mathbf{w}_1, \mathbf{v}_1, \mathbf{w}_2, \mathbf{v}_\Phi\}$ and then this expression is differentiated with respect to $\mathbf{v}_\Phi$. The expression for θ_Φ is given by

$$\theta_\Phi = \mathrm{atan2}(\mathcal{Y}_\Phi, \mathcal{X}_\Phi) + \pi \tag{80}$$

where the terms in Eq. (80) are defined by $\mathcal{Y}_\Phi = \mathbf{v}_2^{*^T}[\mathbf{v}_1\times]\mathbf{v}_\Phi$ and $\mathcal{X}_\Phi = \mathbf{v}_2^{*^T}[\mathbf{v}_1\times]^2\mathbf{v}_\Phi$. The expression for the sensitivity $\frac{\mathrm{d}\theta_\Phi}{\mathrm{d}\mathbf{v}_\Phi^T}$ can now be calculated from Eq. (80). It follows that

$$\frac{\mathrm{d}\theta_\Phi}{\mathrm{d}\mathbf{v}_\Phi^T} = \frac{1}{\mathcal{X}_\Phi^2 + \mathcal{Y}_\Phi^2}\left(\mathcal{X}_\Phi\frac{\partial\mathcal{Y}_\Phi}{\mathrm{d}\mathbf{v}_\Phi^T} - \mathcal{Y}_\Phi\frac{\partial\mathcal{X}_\Phi}{\mathrm{d}\mathbf{v}_\Phi^T}\right) \tag{81}$$

Then the expression for $\frac{\mathrm{d}\theta_\Phi}{\mathrm{d}\mathbf{v}_\Phi^T}$ is evaluated at $\Phi = 0$, resulting in $\mathbf{v}_\Phi = \mathbf{v}_2$ and the derivative terms can be written as $\frac{\partial\mathcal{Y}_\Phi}{\mathrm{d}\mathbf{v}_\Phi^T} = \mathbf{v}_2^{*^T}[\mathbf{v}_1\times]$ and $\frac{\partial\mathcal{X}_\Phi}{\mathrm{d}\mathbf{v}_\Phi^T} = \mathbf{v}_2^{*^T}[\mathbf{v}_1\times]^2$. Using these expressions in Eq. (81) the first sensitivity term can be written as

$$\left.\frac{\mathrm{d}\theta_\Phi}{\mathrm{d}\mathbf{v}_\Phi^T}\right|_{\mathbf{v}_\Phi=\mathbf{v}_2} = \frac{1}{\mathcal{X}^2 + \mathcal{Y}^2}(\mathcal{X}\mathbf{v}_2^{*^T}[\mathbf{v}_1\times] - \mathcal{Y}\mathbf{v}_2^{*^T}[\mathbf{v}_1\times]^2) \tag{82}$$

The expression for $\frac{\mathrm{d}\mathbf{v}_\Phi}{\mathrm{d}\Phi}$ can be determined from Eq. (69):

$$\frac{\mathrm{d}\mathbf{v}_\Phi}{\mathrm{d}\Phi} = (-I_{3\times3}\sin(\Phi) + \cos(\Phi)[\mathbf{v}_1\times]/\|\mathbf{v}_2\times\mathbf{v}_1\|)\mathbf{v}_2 \tag{83}$$

By setting $\Phi = 0$ in Eq. (83), the expression for $\frac{\mathrm{d}\mathbf{v}_\Phi}{\mathrm{d}\Phi}|_{\Phi=0}$ can be determined to be

$$\left.\frac{\mathrm{d}\mathbf{v}_\Phi}{\mathrm{d}\Phi}\right|_{\Phi=0} = \frac{[\mathbf{v}_1\times]\mathbf{v}_2}{\|\mathbf{v}_2\times\mathbf{v}_1\|} \tag{84}$$

Then combining Eq. (82) and (84) the sensitivity of the solution to out-of-plane deflection defined in Eq. (77) can be expressed as

$$\left.\frac{\mathrm{d}\theta_\Phi}{\mathrm{d}\Phi}\right|_{\Phi=0} = \frac{1}{\mathcal{X}^2 + \mathcal{Y}^2}(\mathcal{X}\mathbf{v}_2^{*^T}[\mathbf{v}_1\times] - \mathcal{Y}\mathbf{v}_2^{*^T}[\mathbf{v}_1\times]^2)\frac{[\mathbf{v}_1\times]\mathbf{v}_2}{\|\mathbf{v}_2\times\mathbf{v}_1\|} \tag{85}$$

Using the identity $[\mathbf{v}_1\times]^3 = -[\mathbf{v}_1\times]$ and the definitions of $\mathcal{Y}$ and $\mathcal{X}$ then Eq. (85) becomes

$$\left.\frac{\mathrm{d}\theta_\Phi}{\mathrm{d}\Phi}\right|_{\Phi=0} = \frac{1}{\mathcal{X}^2 + \mathcal{Y}^2}\frac{(\mathcal{X}^2 + \mathcal{Y}^2)}{\|\mathbf{v}_2\times\mathbf{v}_1\|} \tag{86}$$

Then finally by simplifying Eq. (86) the final expression for the sensitivity of the solution to out-of-plane deflection is given by

$$\left.\frac{\mathrm{d}\theta_\Phi}{\mathrm{d}\Phi}\right|_{\Phi=0} = \frac{1}{\|\mathbf{v}_2\times\mathbf{v}_1\|} \tag{87}$$

It is expected that the out-of-plane deflection is small under most operating conditions and therefore Eq. (87) gives a good approximation for the sensitivity of the relative attitude solution due to constraint validation. Note that the sensitivity for out-of-plane deflections is singular for nonobservable configurations, i.e., when $\mathbf{v}_1$ is parallel to $\mathbf{v}_2$.

VII. Simulations

Two simulations scenarios are presented: a static formation and a dynamic configuration of two vehicles, with each vehicle having light source devices and FPDs, which produce a set of parallel LOS measurements. Also each vehicle is observing a common object, other than the other vehicle. As mentioned previously the location of this object is not required for the attitude solution, only the LOS vectors from each vehicle to the object are needed.

A. Static Formation Simulation

The formation configuration uses the following true LOS vectors:

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} \cos(135°) \\ 0 \\ -\sin(135°) \end{bmatrix} \tag{88}$$

The last vector is chosen so that a triangle configuration is assured for the true vectors. The remaining LOS truth vectors are determined from those listed in Eq. (10), without noise added, using the appropriate attitude transformation. For this configuration the true relative attitude is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{89}$$

For the simulation the LOS vectors are converted into focal plane coordinates and random noise is added to the true values having covariances described in Sec. IV, with $\sigma = 17 \times 10^{-6}$ rad. Since each FPD has its own boresight axis, and the measurement covariance in Eq. (27) is described with respect to the boresight, individual sensor frames must be defined to generate the FPD measurements. The measurement error covariance for each FPD is determined with respect to the corresponding sensor frames and must be rotated to the vehicle's body frame as well. The letter S is used to denote sensor frame. The orthogonal transformations for their respective sensor frames, denoted by the subscript, used to orientate the FPD to the specific vehicle, denoted by the superscript, are given by

$$A_{s_{\mathcal{B}_1}}^{\mathbf{v}_1} = \begin{bmatrix} -0.8373 & -0.2962 & 0.4596 \\ -0.2962 & -0.4609 & 0.8366 \\ 0.4596 & -0.8366 & 0.2981 \end{bmatrix}$$
$$A_{s_{\mathcal{B}_1}}^{\mathbf{v}_2} = \begin{bmatrix} -0.8069 & 0.4487 & 0.3843 \\ 0.4487 & -0.0423 & 0.8927 \\ 0.3843 & 0.8927 & -0.2355 \end{bmatrix} \tag{90a}$$

$$A_{s_{\mathcal{B}_2}}^{\mathbf{w}_1} = \begin{bmatrix} -0.8889 & 0.0644 & 0.4535 \\ 0.0644 & -0.9626 & 0.2630 \\ 0.4535 & 0.2630 & 0.8515 \end{bmatrix}$$
$$A_{s_{\mathcal{B}_2}}^{\mathbf{w}_2} = \begin{bmatrix} 0.4579 & -0.0169 & 0.8888 \\ -0.0169 & -0.9998 & -0.0103 \\ 0.8888 & 0.0103 & -0.4581 \end{bmatrix} \tag{90b}$$

The configuration is considered for 1,000 Monte Carlo trials. Measurements are generated in the sensor frame and rotated to the body frame to be combined with the other measurements to determine the full relative attitudes. The wide-FOV measurement model for the FPD LOS covariance is used. Relative attitude angle errors are displayed in Fig. 4. Good performance characteristics are given using the constrained solution. This figure shows that the derived attitude-error covariance does indeed bound these errors in a 3σ sense, which is computed to be

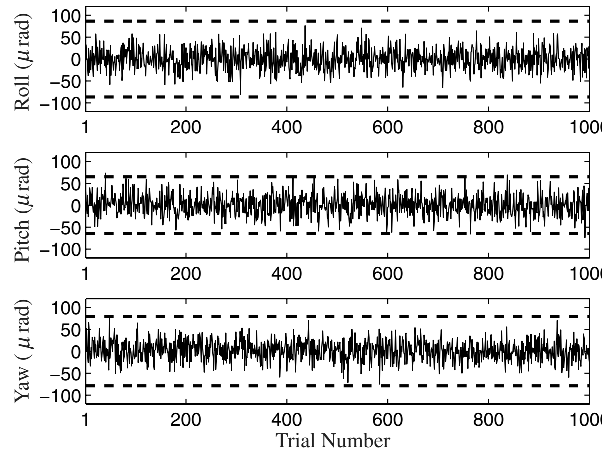

Fig. 4 Relative attitude estimate errors.

$$P_{\delta\alpha\delta\alpha} = 1 \times 10^{-9} \begin{bmatrix} 0.9790 & -0.1166 & -0.0889 \\ -0.1166 & 0.4056 & 0.0889 \\ -0.0889 & 0.0889 & 0.5308 \end{bmatrix} \tag{91}$$

This configuration is also considered for 100 Monte Carlo trials for various out-of-plane deflection angles. The angle is varied from −0.05 to 0.05 deg using 0.01 deg intervals. Measurements are generated in the sensor frame and rotated to the body frame to be combined with the other measurements to determine the full relative attitudes. The wide-FOV measurement model for the FPD LOS covariance is used. The Monte Carlo relative attitude angle errors are calculated for each trial and are plotted for all the considered out-of-plane deflection angles. The covariance of the angle error is calculated for out-of-plane deflection angles given the 100 Monte Carlo runs. The numerical variance runs are plotted with the angle errors for all deflection angles. Results are shown in Fig. 5. The theoretical error is calculated using Eq. (87) and the linear approximation about $\Phi = 0$, given by $\Theta = \frac{d\Theta}{d\Phi}\big|_{\Phi=0}\Phi$. Good agreement between computed errors through the Monte Carlo runs and the theoretical predictions is shown. Also the numerical variance does not vary with out-of-plane deflection; moreover, out-of-plane deflection biases the solution only and does not increase its variation about the mean.

B. Dynamic Formation Simulation

In the dynamic configuration two vehicles are flying over a tracked object, with unknown position, as they measure LOS vectors to each other and the object in the formation. In this configuration it is

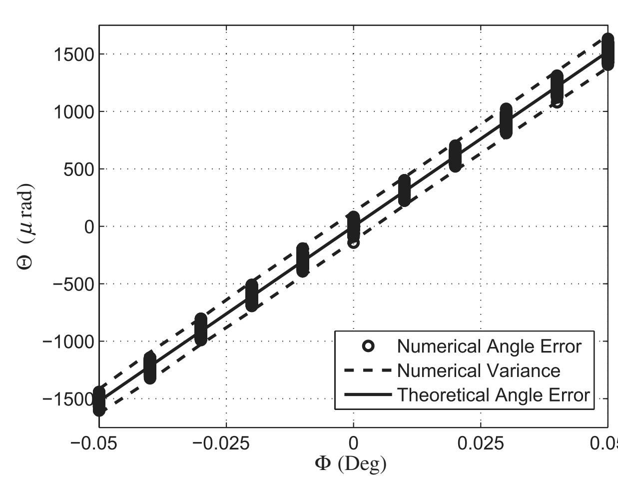

Fig. 5 Sensitivity to out-of-plane deflection.

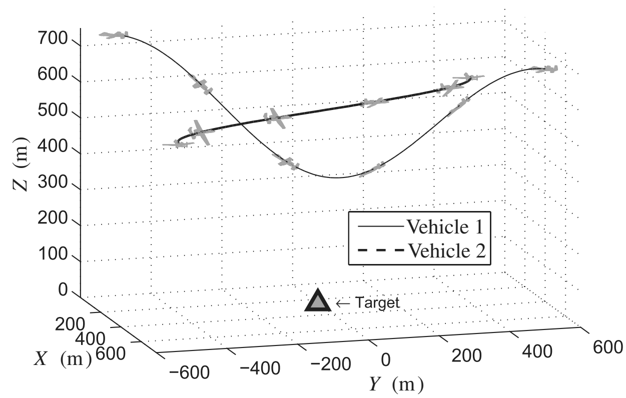

Fig. 6 Dynamic trajectories.

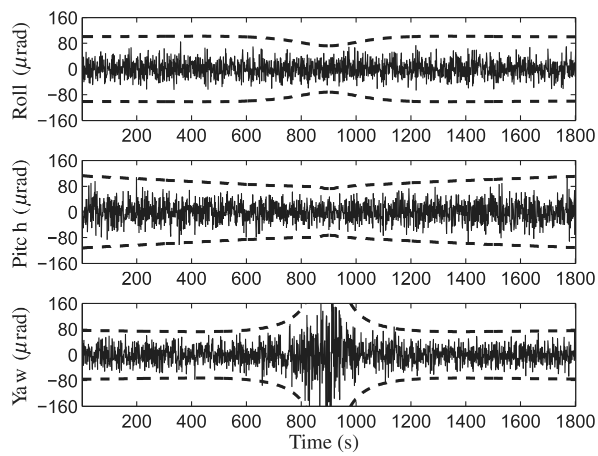

Fig. 7 Relative attitude errors for dynamic configuration.

assumed that the FPD devices are gimballed allowing constant visibility and, for simplicity, it is assumed that the rotation matrix between the body frames to the sensor frames are those listed for the static simulations. The aircraft trajectories are displayed in Fig. 6. The vehicles average airspeed is approximately 60 (km/h) and the LOS vectors are sampled at 0.1 Hz for a total of 180 s. LOS vectors are converted into focal plane coordinates and random noise is added to the true values having covariances described in Sec. IV, with $\sigma = 17 \times 10^{-6}$ rad. The algorithm in Sec. III is used to provide a point-by-point solution for the relative attitude. The relative attitude that is determined solves the transformation from the $\mathcal{B}_1$ frame to the $\mathcal{B}_2$ frame. These frames are defined as: the X body axis is positive along forward and positive roll is a clockwise rotation as viewed from behind the vehicle; the Y body axis is positive along the right wing and positive pitch is defined as a nose up rotation; and the Z body axis is positive down in Fig. 6 and positive yaw is defined as clockwise rotation as viewed from above the vehicle.

Figure 7 displays the relative attitude errors for the dynamic configuration. The magnitude of the relative attitude errors dependence on geometry can clearly be seen. As the LOS geometry changes throughout the trajectory, the 3σ bounds on the errors also change and accurately bound the estimated attitude errors. A large increase in the relative yaw error can be seen as the LOS configuration approaches an extreme condition where $\mathbf{w}_1$, $\mathbf{w}_2$ and $A_{\text{true}}\mathbf{v}_2$ are nearly parallel. This results in a near unobservable situation, which is correctly depicted in the covariance.

VIII. Conclusions

In this paper a new relative attitude determination approach for two vehicles was presented. The approach requires LOS information between each vehicle and LOS information to a common object. This solution provides a point-by-point solution for the relative attitude of a two-vehicle formation. The advantage of this approach is that the object's position does not need to be known at all. The ambiguity present in the unconstrained solution was removed and the observability issues resolved. The heart of the approach relies on the notion that all vectors form a triangle, which was used as a constraint in the developed solution. In actual practice, the triangle scenario reflects a realistic physical situation especially for large distances between vehicles and the common object. Still, an out-of-plane vector may exist due to sensor errors, such as misalignments. The analytically derived expression for the sensitivity of out-of-plane errors is useful to quantify whether or not a particular system using the constrained triangle solution causes issues in comparison to the required accuracy of the estimated solution. Both static and dynamic application of the solution with their 3σ bounds showed that the solution was found to have good performance characteristics.

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